CONNEXIVE IMPLICATION, MODAL LOGIC AND SUBJUNCTIVE CONDITIONALS *

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Professor Storrs McCall and I share an interest in logical systems which contain the non-classical theorems:

1. $-(p\rightarrow -p)$ - It is false that if p then not-p.

2. $(p\rightarrow q)\rightarrow -(p\rightarrow -q)$ - If (if p then q) then it is false that (if p then not-q); although our motives differ. I called my system, PA1,(JSL,1962) a logic of subjunctive conditionals; he called his system, CC1,(JSL,1966) a system of "connexive implication" and allied himself with those who, according to Sextus Empiricus, "say that a conditional is sound when the contradictory of its consequent is incompatible with its antecedent". Nevertheless, our two systems are very closely related formally. McCall, in effect, added five axioms to my system and established the completeness of this expanded axiom-set with respect to the same truth-tables I had used in 1962 for the primitives, the conditional, conjunction and negation. PA1 showed it possible to have a consistent propositional logic which 1) contains the classical PM calculus (with '$\supset$' interpreted as 'not..or..'), 2) eliminates all the so-called paradoxes of material and strict implication from the conditional, 3) includes most of the traditional logical principles involving conditionals, and yet 4) includes the non-classical theorems mentioned above. McCall pointed out the independence of all such systems of any of the well-known systems of logic and proved his system Post-complete.

Besides these two systems there are many other constructible systems which share the properties just described. The problem is to find a satisfactory one. Certain difficulties of interpretation arose in connection with PA1 which led me to look for better systems; these difficulties are aggravated, rather than modified, by the new axioms in McCall's expansion, although from a formal point of view his system is certainly the more interesting. These difficulties, as well as the relationships of these two systems to each other and to modal logic, stand out clearly in the light of an observation which McCall credits to Meredith - namely, that the truth-table we both used for the conditional can be eliminated in favor of a unary modal operator.

In this paper I present two modal logics, PA1m and CC1m, which use C.I.Lewis's primitives for possibility, negation and conjunction, and Lewis's definitions of other logical constants, but yield respectively my so-called "logic of subjunctive conditionals" and McCall's system of "connexive implication". The four-valued truth-tables for negation and possibility are those of Lewis's Group II matrices; the truth-table for conjunction is that of PA1 and CC1, not Lewis's. On this basis, the defined conditional comes out to have the same truth-table as that assigned in PA1 and CC1. This suggests the odd conclusion that the difference between PA1 and CC1 on the one hand and Lewis's systems was not related to conditionality or possibility so much as to the different concepts of conjunction.

Table I shows three axiom sets: Lewis's S3, the modal version, PA1m, of my logic of subjunctive conditionals and a modal version, CC1m, of McCall's system of "connexive implication". The matrices establish the consistency of the various systems presented, and the derivations appended to this paper show that CC1m and PA1m are complete with respect to McCall's CC1, and my PA1, respectively. Table II shows the interrelationships between the axioms and theorems of Lewis's systems, S1,S2,S3,S4,S5, McCall's CC1, my PA1, PA1m, CC1m and Rosser's axiomatization of of the classical propositional calculus of *Principia Mathematica*. The following remarks draw together and reflect upon some of the results shown in these two tables.

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The differences between S3 and PAlm are not so great, in one respect, as they first appear to be. The formulas appearing as A1 and A5 in each are mutually derivable in the other; axioms 4,6, and 7 are identical in both systems. Thus, the real differences boil down to the fact that the strict implications in S3's axioms 2 and 3 are merely the corresponding truth-functional conditionals in PAlm, and that PAlm contains, in Axiom 8 the non-classical formula, $(p\rightarrow q)\rightarrow -(p\rightarrow -q)$.

Examination of <u>Table II</u> shows that in McCall's and my systems the formulae 3, $(q.p)\rightarrow p$ and 6. $p\rightarrow(p.p)$ are nowheres derivable. As Everett Nelson pointed long ago, the non-derivability of the first of these, Simplification, is a price we must pay for using the non-classical theorems with standard transposition, syllogism, and the ordinary rules of substitution. Both McCall's system and mine must face up to the demand that we either revise our systems to include these theorems, or explain why they are non-derivable and justify their non-inclusion. In modal logic, the non-derivability of these formula leads to the non-inclusion of the distinctive axiom of S2, $\Diamond(q.p)\rightarrow\Diamond p$. [illegible] But $\Diamond(q.p)\supset\Diamond p$ also fails in all those systems; and this is clearly due to the properties of conjunction (as relfected in the different conjunction matrix).

Secondly, all of McCall's and my systems include, as intended, the following theroems or axioms which are not derivable in Lewis or in classical logic:

27. $(p\rightarrow q)\rightarrow -(p\rightarrow -q)$
28. $(p\rightarrow p)\rightarrow -(p\rightarrow -p)$
29. $-(p\rightarrow -p)$

and others. Ordinarily, this would be cause for self-congratulation. But when these theorems are reduced to modal propositions they present serious problems of interpretation. They become, respectively, equivalent to theorems stating:

27'. $\Diamond(p.q)\rightarrow\Diamond(p.-q)$
28'. $\Diamond(p.p)\rightarrow\Diamond(p.-p)$
29'. $\Diamond(p.p)$

The first would seem false whenever q is a tautology; the second would seem false whenever p is consistent; and the third would be false whenever p wast inconsistent. These consequences alone seem fairly devastating for both of our systems.

When we consider the nex axioms McCall added to establish Post-completeness, however, the difficulties in interpretation increase. To be sure some of additional axioms in CC1 (cf. 20,21,22,23,and 26 in <u>Table II</u>) seem plausible, e.g., his CC1 Axiom 2, $(((p\rightarrow p)\rightarrow q)\rightarrow q)$ (#20 in Table II) which occurs in all Lewis systems except [illegible], as well as in PM. But there is a peculiar and irrational bias in some of them. Thus Axiom 7 of CC1 (#23 in Table II), $p\rightarrow((p.p).p)$, seems eminently plausible until it is realized that while p will imply (or be implied by) any conjunction containing just an odd number of iterations of itself, it never implies (or is implied by) a conjunction with an even number of conjuncts of itself. Thus

$p\rightarrow((p.p).p)$
$p\rightarrow((((p.p).p).p).p)$
$p\rightarrow((((((p.p).p).p).p).p).p)$

are all logical truths, but

$p\rightarrow(p.p)$
$p\rightarrow(((p.p).p).p)$
$p\rightarrow(((((p.p).p).p).p).p)$

are merely contingent. Again, any conjunction containing just an even number of conjuncts of p, as in

$(p.p)\rightarrow(q\rightarrow q)$ (CC1 Ax 4, #21 in Table II)
$((p.p).(p.p))\rightarrow(q\rightarrow q)$, etc.

will imply any theorem of CC1 , but no conjunctions having just an odd number of occurrences of a given variable will imply any theorem. Similar remarks pertain to to the double p's in CCl Axiom 7 (#22 in Table II). It is hard to how the concept of connexive implication - that logically true conditionals have antecedents which are cimpatible with the contradictories of their consequents,- can justifiy these distinctions between odd and even numbers of variable occurrences. In the Modal versions of CCl, the modal correlates of these implausibilities (cf. 35,36,37,38 in Table II, which represent the Axioms ~~of~~ (except Ax 8) of CClm which differ from those of PAlm) seem just as unlikely:

35. $\Box\Diamond(p.p)$ (Axiom 1, CClm)
36. $-\Diamond(p.p)\rightarrow(q\rightarrow q)$ (Axiom 2, CClm)
37. $(p.p)\rightarrow\Box(p.p)$ (Axiom 9, CClm)
38. $(p.-\Box p)\Rightarrow((q\vee q)\rightarrow p)$ (Axiom 10, CClm)

The only one of these four that is included in the modal systems of Lewis is the second, #36, and this only because it is a paradox of strict implication. The other three are not derivable in any of the five Lewis systems, and in any case are intuitively unconvincing. The peculiarities of a type of conjunction which yields different implications for odd-numbered conjunctive iterations of a variable than for even-numbered ones stand out in all four of these; each fails if an even-numbered conjunctive iteration (or alternation) is replaced by an odd-numbered one. These same peculiarlities are reflected in the difficulty of finding a consistent interpretation for the conjunction matrix axiomatized in CCl, PAl etc.

Among the interesting formal results in McCall's system is that fact that not only can we define connexive implication in terms of negation, conjunction and a modal operator (possibility or necessity), but we can define the modal operators (either possibility or necessity) in terms of this primitives of CCl, i.e., negation, conjunction and the conditional. Thus we could have, in CCl, the definitions:

$$\Box p =_{df} ((p\rightarrow p)\rightarrow p) \qquad \Diamond p =_{df} -((-p\rightarrow -p)\rightarrow -p)$$

Since McCall proved that all tautologies are theorems in hisbsystem, and the matrices for these defined terms are identical with those already referred to (the Group II Lewis matrices), it follows that the nine axioms of CClm can also be used for CCl with the conditional primitive instead of the modal operator. Thus CCl and CClm are exactly equivalent systems. Since CCl was proved Post-complete, CClm can be proved Post-complete also. Since CCl is functionally incomplete, CClm is functionally in complete also.

In spite of the instructive and interesting formal properties revealed in CCl and PAl, in my opinion, the foregoing analysis shows rather conclusively the inadequacy as a formalization of a viable logic, of both my system PAL and McCall's CCl. Admitting these inadequacies do not, of course, entail rejection of the non-classical theoriems. There are other systems which contain these theorems and lack the objectionalbe features just discussed. Although not fully satisfactory systems, PAl and CCl are, I believe, helpful first efforts towards the construction of a satisfactory non-classical logic.

TABLE I - A COMPARISON OF S-3, PAl(Modal) AND CCl(Modal)

I. Primitive Symbols

1. Grouping devices: ( )
2. Logical Constants: $-$ $.$ $\Diamond$
3. Propositional variables: p q r s p' ...

II. Rules of Formation

F1. A single variable, by itself, is wf.
F2. If S is wf, then $\ulcorner -S \urcorner$ and $\ulcorner \Diamond S \urcorner$ are wf.
F3. If S and S' are wf, then $\ulcorner (S.S') \urcorner$ is wf.

III. Abbreviations (Definitions)

D1. $\ulcorner (S \vee R) \urcorner$ for $\ulcorner -(-S.-R) \urcorner$
D2. $\ulcorner (S \supset R) \urcorner$ for $\ulcorner -(S.-R) \urcorner$
D3. $\ulcorner (S \equiv R) \urcorner$ for $\ulcorner ((S \supset R).(R \supset S)) \urcorner$
D4. $\ulcorner (S \rightarrow R) \urcorner$ for $\ulcorner -\Diamond(S.-R) \urcorner$
D5. $\ulcorner (S \leftrightarrow R) \urcorner$ for $\ulcorner ((S \rightarrow R).(R \rightarrow S)) \urcorner$
D6. $\ulcorner \Box S \urcorner$ for $\ulcorner -\Diamond -S \urcorner$

IV. Rules of Transformation (Rules of Inference)

R1. If $\vdash S$ and $\vdash \ulcorner (S \rightarrow R) \urcorner$ then $\vdash R$. (Modus Ponens)
R2. If $\vdash S$ and $\vdash R$ then $\vdash \ulcorner (S.R) \urcorner$ (Adjunction)
R3. If $\vdash S$ and $\vdash S'$ is formed from S by substituting some wff at every occurrence of a propositional variable in S, then $\vdash S'$. (Substitution)
*R4. If $\vdash (S \leftrightarrow R)$ and $\vdash Q$, then if Q' is formed by replacing an occurrence of S in Q by R, then $\vdash Q'$. (Rule of Replacement of Strict Equivalents)

Matrices for consistency proofs.

$-$	p		$\Diamond$	p
4	1		1	1
3	2		2	2
2	3		1	3
1	4		4	4

(Satisfied in S3, PAlm and CClm)

(p.q)	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

(Satisfied in S3, not others)

(p.q)	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	3	4
4	4	3	4	3

(Satisfied in PAlm, CClm only)

S3 Axioms (Lewis & Langford, 1932, p493)

A1. $(p.q) \rightarrow (q.p)$
A2. $(q.p) \rightarrow p$
A3. $p \rightarrow (p.p)$
A4. $(p.(q.r)) \rightarrow (q.(p.r))$
A5. $((p \rightarrow q).(q \rightarrow r)) \rightarrow (p \rightarrow r)$
A6. $-\Diamond p \rightarrow -p$
A7. $(p \rightarrow q) \rightarrow (-\Diamond q \rightarrow -\Diamond p)$

PAlm Axioms (Angell, JSL, 1962)

A1. $p \rightarrow p$
A2. $(q.p) \supset p$
A3. $p \supset (p.p)$
A4. $(p.(q.r)) \rightarrow (q.(p.r))$
A5. $((r.p).-(q.r)) \rightarrow (p.-q)$
A6. $-\Diamond p \rightarrow -p$
A7. $(p \rightarrow q) \rightarrow (-\Diamond q \rightarrow -\Diamond p)$
A8. $(p \rightarrow q) \rightarrow -(p \rightarrow -q)$

CClm Axioms (McCall, JSL, 1966)

A1. $\Box\Diamond(p.p)$
A2. $--\Diamond(p.p) \rightarrow (q \rightarrow q)$
A3. $p \supset (p.p)$
A4. $(p.(q.r)) \rightarrow (q.(p.r))$
A5. $((r.p).-(q.r)) \rightarrow (p.-q)$
A6. $-\Diamond p \rightarrow -p$
A7. $(p \rightarrow q) \rightarrow (-\Diamond q \rightarrow -\Diamond p)$
A8. $p \rightarrow ((p.p).p)$
A9. $(p.p) \rightarrow \Box(p.p)$
A10. $(p.-\Box p) \supset ((q \vee q) \rightarrow p)$

*McCall, JSL, Sept 1966, proves that this rule is derivable in CCl, using theorems which can be established in S3, PAl and CCl, though not in S1 or S2. Hence R4 is derivable in all three systems.

TABLE II - THE INTERRELATIONSHIPS OF AXIOMS IN S1,S2,S3,S4,S5,PA1,PA1m,CC1,CC1m,PM(Rosser)

		S1	S2	S3	S4	S5	PA1	PA1m	CC1	CC1m	PM
	Classical Theorems										
1.	$p \rightarrow p$	+	+	+	+	+	*7	Ax 1	*23	*2	+
2.	$(p.q) \rightarrow (q.p)$	Ax 1	Ax 1	Ax 1	Ax 1	Ax 1	*20	*4	*27	*4	+
3.	$(q.p) \rightarrow p$	Ax 2	Ax 2	Ax 2	Ax 2	Ax 2	-	-	-	-	+
4.	$(q.p) \supset p$	+	+	+	+	+	+	Ax 2	+	*83	+
5.	$(p.q) \supset p$	+	+	+	+	+	Ax 8	*73	*92	*84	Ax 2
6.	$p \rightarrow (p.p)$	Ax 3	Ax 3	Ax 3	Ax 3	Ax 3	-	-	-	-	+
7.	$p \supset (p.p)$	+	+	+	+	+	Ax 9	Ax 3	Ax 10	Ax 3	Ax 1
8.	$(p.(q.r)) \rightarrow (q.(p.r))$	Ax 4	Ax 4	Ax 4	Ax 4	Ax 4	Ax 4	Ax 4	Ax 5	Ax 4	+
9.	$((p \rightarrow q).(q \rightarrow r)) \rightarrow (p \rightarrow r))$	Ax 5	Ax 5	Ax 5	Ax 5	Ax 5	+	*66	+	*66	+
10.	$((p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)))$	-	-	+	+	+	*43	*44	Ax 1	*44	+
11.	$((q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$	-	-	+	+	+	Ax 1	*31	*24	*31	+
12.	$((r.p).-(q.r)) \rightarrow (p.-q)$	+	+	+	+	+	+	Ax 5	+	Ax 5	+
13.	$(p \supset q) \supset (-(q.r) \supset -(r.p))$	+	+	+	+	+	*74	*67	*94	*67	Ax 3
14.	$(p \rightarrow q) \rightarrow ((r.p) \rightarrow (q.r))$	-	+	+	+	+	Ax 2	*3	+	*3	+
15.	$(p \rightarrow -(q.r)) \rightarrow ((q.p) \rightarrow -r)$	+	+	+	+	+	Ax 3	*59	+	*59	+
16.	$((p.q) \rightarrow r) \rightarrow ((p.-r) \rightarrow -q)$	+	+	+	+	+	+	+	*160	+	+
17.	$(p \rightarrow -q) \rightarrow (q \rightarrow -p)$	+	+	+	+	+	Ax 5	*35	*40	*35	+
18.	$--p \rightarrow p$	+	+	+	+	+	Ax 6	*7	*64	*7	+
19.	$(p \rightarrow q) \rightarrow (p \supset q)$	+	+	+	+	+	Ax 7	*1	*89	*73	+
20.	$((p \rightarrow p) \rightarrow q) \rightarrow q$	-	+	+	+	+	-	-	Ax 2	*94	+
21.	$(q.q) \rightarrow (p \rightarrow p)$	+	+	+	+	+	-	-	Ax 4	*96	+
22.	$(p.p) \rightarrow ((p \rightarrow p) \rightarrow (p.p))$	-	-	-	-	-	-	-	Ax 6	*112	+
23.	$p \rightarrow ((p.p).p)$	+	+	+	+	+	-	-	Ax 7	Ax 8	+
24.	$((p \rightarrow -q).q) \rightarrow -p$	+	+	+	+	+	+	*64	Ax 8	*64	+
25.	$(p.-(p.-q)) \rightarrow q$	+	+	+	+	+	+	*65	Ax 9	*65	+
26.	$((-p \vee ((p \rightarrow p) \rightarrow p)) \vee (((p \rightarrow p) \vee (p \rightarrow p)) \rightarrow p)$	-	-	-	-	-	-	-	Ax 11	*121	+
	Non-Classical Theorems										
27.	$(p \rightarrow q) \rightarrow -(p \rightarrow -q)$	-	-	-	-	-	Ax 10	Ax 8	*102	+	-
28.	$(p \rightarrow p) \rightarrow -(p \rightarrow -p)$	-	-	-	-	-	+	*2	Ax 12	*108	-
29.	$-(p \rightarrow -p)$	-	-	-	-	-	*77	+	+	+	-
	Modal Theorems										
30.	$-\Diamond p \rightarrow -p$	Ax 6	Ax 6	Ax 6	Ax 6	Ax 6		Ax 6		Ax 6	
31.	$\Diamond(p.q) \rightarrow \Diamond p$	-	Ax 7	+	+	+		-		-	
32.	$(p \rightarrow q) \rightarrow (-\Diamond q \rightarrow -\Diamond p)$	-	-	Ax 7	+	+		Ax 7		Ax 7	
33.	$\Diamond\Diamond p \rightarrow \Diamond p$	-	-	-	Ax 7	+		-		+	
34.	$\Diamond p \rightarrow \Box\Diamond p$	-	-	-	-	Ax 7		-		-	
35.	$\Box\Diamond(p.p)$	-	-	-	-	-				Ax 1	
36.	$--\Diamond(q.q) \rightarrow (p \rightarrow p)$	+	+	+	+	+				Ax 2	
37.	$(p.p) \rightarrow \Box(p.p)$	-	-	-	-	-				Ax 9	
38.	$(p.-\Diamond p) \supset ((q \vee q) \rightarrow p)$	-	-	-	-	-				Ax 10	

Notes:

'+' means that the theorem to the left is derivable in the system indicated above it.

'-' means that the theorem to the left is provably not derivable in this system.

S1,S2,S3, S4, S5, are based on the formulations in Feys,R, Modal Logics,1965, except that '$p \rightarrow \Diamond p$' and '$(p \rightarrow q) \rightarrow (\Diamond p \rightarrow \Diamond q)$' are replaced by the axioms '$-\Diamond p \rightarrow -p$' and '$(p \rightarrow q) \rightarrow (-\Diamond q \rightarrow -\Diamond p)$' in Lewis and Langford, 1932, p493.

PA1, refers to the system of Angell, JSl, Sept 1962.

CC1 refers to the system of connexive implication in McCall, JSL, Sept 1966.

PA1m and CC1m are the modal versions of PA1 and CC1.

'*n' gives the number of the theorem in the system indicated at the top of the column.

PA1$_{14}$(Modal)

A1. $p\to p$
A2. $(q.p)\supset p$
A3. $p\supset(p.p)$ — CC1 Ax 10; PA1 Ax. 9
A4. $(p.(q.r))\to(q.(p.r))$ — CC1 Ax. 5; PA1 Ax. 4
A5. $((r.p).-(q.r))\to(p.-q)$
A6. $-\Diamond p\to -p$
A7. $(p\to q)\to(-\Diamond q\to -\Diamond p)$
A8. $(p\to q)\to -(p\to -q)$ — PA1- Ax. 10

1. [A6 - 13] $(p\to q)\to -(p.-q)$ — PA1 Ax. 7
2. [A8 = 2] $(p\to p)\to -(p\to -p)$ — CC1 Ax. 12
3. [A7 = (A5 - 3] $(p\to q)\to((r.p)\to(q.r))$ — [PA1 Ax. 2]
4. [3 = (A1 - 4] $(p.q)\to(q.p)$ — [Lewis S3, Ax 1]
5. [A7 = (4 -5] $-\Diamond(q.p)\to -\Diamond(p.q)$
6. [D4, 5 = 6] $(-q\to p)\to(-p\to q)$
7. [6 = (A1 - 7] $--p\to p$ — [PA1 Ax. 6]
8. [3 = (7 - 8] $(q.--p)\to(p.q)$ — (Also in S3, 1932, but shown
9. [A7 = (8 - 9] $(p\to q)\to(-q\to -p)$ — derivable by McKinsey, 1935)
10. [9 = (4 -10] $-(q.p)\to -(p.q)$ — CC1 Ax. 9
11. [D4, 5 =(A1 - 11] $-\Diamond(-p.p)$
12. [A7 = (8 - 12)] $-\Diamond(-p.p)\to -\Diamond(p.---p)$
13. [12 = (11 - 13)] $p\to --p$
14. [3 = (4 - 14)] $(-(q.r).(p.r))\to((r.p).-(q.r))$
15. [A7 = (14 - 15)] $-\Diamond((r.p).-(q.r))\to -\Diamond(-(q.r).(p.r))$
16. [9 = (15 - 16)] $--\Diamond(-(q.r).(p.r))\to --\Diamond((r.p).-(q.r))$
17. [3 = (16- 17)] $((p\to q).--\Diamond(-(q.r).(p.r)))\to(--\Diamond((r.p).-(q.r)).(p\to q))$
18. [A7= (17 - 13)] $-\Diamond(--\Diamond((r.p).-(q.r)).(p\to q))\to -\Diamond((p\to q).--\Diamond(-(q.r).(p.r)))$
19. [5 = (3 - 19)] $-\Diamond(--\Diamond((r.p).-(q.r)).(p\to q))$
20. [18 = (19 - 20)] $-\Diamond((p\to q).--\Diamond(-(q.r).(p.r)))$
21. [5 = (20 - 21)] $-\Diamond(--\Diamond(-(q.r).(p.r)).(p\to q))$
22. [9 = (5 - 22)] $--\Diamond(q.p)\to --\Diamond(p.q))$
23. [3 = (22 - 23)] $(r.--\Diamond(q.p))\to(--\Diamond(p.q).r)$
24. [A7 = (23 - 24) $-\Diamond(--\Diamond(p.q).r)\to -\Diamond(r.--\Diamond(q.p))$
25. [D4, 22= (21 -25)] $(p\to q)\to((p.r)\to(q.r))$
26. [9 = (A7 - 26)] $-(-\Diamond q\to -\Diamond p)\to -(p\to q)$
27. [3 = (26 - 27)] $(r.-(-\Diamond q\to -\Diamond p))\to(-(p\to q).r)$
28. [A7 =(27 -28)] $-\Diamond(-(p\to q).r)\to -\Diamond(r.-(-\Diamond q\to -\Diamond p))$
29. [D4, 5 = (25 - 29)] $-\Diamond(-((p.r)\to(q.r)).(p\to q))$
30. [28 = (29 -30)] $-\Diamond((p\to q).-(-\Diamond(q.r)\to -\Diamond(p.r))$
31. [D4, 30 = 31] $(p\to q)\to((q\to r)\to(p\to r))$ — CC1 Ax. 1
32. [31 =(13 - 32)] $(--p\to q)\to(p\to q)$
33. [31 = 6-(32-33))] $(-q\to -p)\to(p\to q)$
34. [31 = (7 - 34)] $(p\to q)\to(--p\to q)$
35. [31 = (34-(33-35))] $(p\to -q)\to(q\to -p)$ — PA1 Ax. 5
36. [31 = 36] $(-q\to -p)\to((-p\to -r)\to(-q\to -r))$
37. [31 = (9-(36-37))] $(p\to q)\to((-p\to -r)\to(-q\to -r))$
38. [31 = (9 - 38)] $((-p\to -r)\to(q\to r))\to((r\to p)\to(-q\to -r))$
39. [31 = (37-(38-39))] $(p\to q)\to((r\to p)\to(-q\to -r))$
40. [31 = (9-(39-40))] $(p\to q)\to(-(-q\to -r)\to -(r\to p))$
41. [9+ (33 - 41)] $-(r\to q)\to -(-q\to -r)$
42. [31= (41 -42)] $(-(-q\to -r)\to -(r\to p))\to(-(r\to q)\to -(r\to p))$
43. [31 = (42-(40-43))] $(p\to q)\to(-(r\to q)\to -(r\to p))$
44. [31 = (41-(33-44))] $(p\to q)\to((r\to p)\to(r\to q))$ — PA1 Ax. 1

45. [44 = (4-45)] ((r.p)→(q.r))→((r.p)→(r.q.))
46. [44 = (45-(3-46))] (p→q)→((r.p)→(r.q)
47. [31 = (4-47)] ((r.p)→(r.q))→((p.r)→(r.q))
48. [31 = (46-(47-48))] (p→q)→((p.r)→(r.q)) CCl Ax. 3
49. [46 = (7 -49)] ((q.p).--r)→((q.p).r)
50. [31= (49-(4-50)) ((q.p).--r)→(r.(q.p))
51. [31 = (50-(A4 - 51))] ((q.p).--r)→(q.(r.p))
52. [46 = (4-52)] (q.(r.p))→(q.(p.r))
53. [31 = (52 -(A4-53))] (q.(r.p))→(p.(q.r))
54. [31 = (51-(53-54))] ((q.p).--r)→(p.(q.r))
55. [44 = (13 - 55)] (p.(q.r))→(p.--(q.r))
56. [31 = (54-(55-56))] ((q.p).--r)→(p.--(q.r))
57. [A7 = (56 -57)] (p→-(q.r))→((q.p)→-r) PAl Ax. 3
58. [44 = (7 -58)] (p→--q)→(p→q)
59. [31 = (57-(58-59))] (p→-(q.-r))→((q.p)→r)
60. [31 = (4 -60)] ((q.p)→r)→((p.q)→r)
61. [31 = (59-(60-61))] (p→-(q.-r))→((p.q)→r)
62. [44= (A6 -62)] (p→--(q.-r))→(p→-(q.-r))
63. [31 = (62-(61-63))] (p→(q→r))→((p.q)→r) **Importation**
64. [63 = (35 - 64)] ((p→-q).q)→-p CCl Ax. 8
65. [59 = (A1 -65)] (p.-(p.-q))→q CCl Ax. 9
66. [63 = (31 -66)] ((p→q).(q→r))→(p→r) S3 Ax. 5
67. [44 = (8-(A5 - 67))] (-(q.r).--(r.p))→(p.-q)
68. [9 = (67 -68)] (p⊃q)→(-(q.r)⊃-(r.p))
69. [A6 = (68-69)] (p⊃q)⊃(-(q.r)⊃-(r.p)) Rosser PM Ax. 3
70. [31 = (53-(4-70))] (p.(q.r))→((p.q).r)
71. [25 = (4 - 71)] ((p.q).-p)→((q.p).-p)
72. [9 = (71-72)] -((q.p).-p)→((p.q).-p)
73. [72 = (A2 - 73)] -((p.q).-p) Rosser PM Ax. 2

CClm (Modal)*

A1. □◇(p.p)
A2. --◇(p. p)→(q→q)
A3. p ⊃ (p.p)
A4. (p.(q.r))→(q.(P.r))
A5. ((r.p).-(q.r))→(p.-q)
A6. -◇p→-p
A7. (p→q)→(-◇q→-◇p)
A8. p→((p.p).p)
A9. (p.p)→N(p.p)
A10. (p.-□p)⊃ (qvq)→p)

The theorems of PAlm, and their justifications, may be kept provided we replace theorems 1 and 2 by

1 [A6 = (A1 - 1)] --◇(q.q)
2 [A2 = (1 - 2)] (p→p),

changing 'A1 to '2' in the proofs of theorems 4, 7, and 11, and replacing theorem 73 in PAlm by what was 2 in PAlm, i.e.,

73 [A6 = 73] (p→q)→-(p.-q) PA1 Axiom 7

The proof of theorem 2 in PAlm does not hold, so far, in CClm since '(p→Q)→-(p→-q)' must be derived from CClm Axiom 12, (p→p)→ -(p→-p) which is derivable in proofs in CClm then may proceed as follows, establishing PAlm's Axiom 2, '(q.p)⊃ p', in theorem 83.

74 [44 = (46 - (33-74))] (p→q)→((r.-q)→(r.-p))
75 [31 = (74 - (9 - 75))] (p→q)→((r⊃ q)→(r⊃p))
76 ~~[31 = (A2-(4-76))] (p.-p) -(q.q)~~ Correction: 76A. [35 = (A6 → 76A)] (q.q)→--◇(q.q)
77 ~~[35 = (76 - 77)] (q.q)→-(p.-p)~~ 76 [31 = (76A→(A2-76)] (q.q)→ (p→p)
77 [31 = (76-(73-77))] (q.q)→ -(p.-p)
78 [61 = (77 - 78)] ((q.q).p)→p)
79 [25 = (4 - 79)] ((r.p).-(q.r))→((p.r).-(q.r))
80 [31 = (A5 - (79 -80))] ((p.r).-(q.r))→(p.-r)
81 [9 = (80 -81)] (p⊃q)→((p.r)⊃ (q.r))
82 [81= (A3 - 82)] (q.p)⊃ ((q.q).p)
83 [75= (78 -(82-83))] (q.p)⊃ p PAlm Axiom 2,S3 Axiom 2
84 [72 = (83-84)] (p.q)⊃ p Rosser, PM Axiom 2
85 [51 = (10 -85)] (q.-(p.q))→-p)
86 [A7 = (85 - 86)] □p→(q→(p.q))
87 [31 = (A6 - (7 -87))] □p→p
88 [31 = (A7 - (33 - 88))] (p→q)→(□p→□q)
89 [31 = (13 - (A2 - 89)] ◇(q.q)→(p→p)
90 [88 = (89 -90)] □◇(q.q)→□(p→p)
91 [90 = (A1 - 91)] □(p→p)
92 [86 = (91 -92)] q→((p→p).q)
93 [6 = (92 - 93)] (p→p)⊃ q)→q
94 [31= (73-(93 - 94))] (((p→p)→q)→q) CCl Axiom 2
95 [33 = (A6 - 95))] p→◇p
96 [31 = (95(A7-96))] (q.q)→(p→p) CCl Axiom 4
97 [31 = (9-(7 - 97))] (-r→(p.-q))→((p⊃q)→r)
98 [31 = (75 -98)] (((p⊃q)→r))→((p→q)→r)
99 [31 = (86-(97-99))] Np→((p⊃q)→q))
100 [31 = (99-(98-100))] Np→((p→q)→q)

* The A2 and A6 may be replaced by A2' ◇(p .p)→(q→q), and A6' □p→p; the form above is utilized simply to preserve similarities between S3 and PAlm and CClm.

101 [100 = (A1 - 101)] $(\Diamond(p.p)\rightarrow\Diamond(p.p))\rightarrow\Diamond(p.p)$

102 [31 = (A7 -(33 - 102))] $(p\rightarrow q)\rightarrow(\Diamond p\rightarrow\Diamond q)$

103 [31 = (3 - (102 -103))] $(p\rightarrow p)\rightarrow(\Diamond(p.p)\rightarrow\Diamond(p.p))$

104 [31 = (103 -(101-104))] $(p\rightarrow p)\rightarrow(p.p)$

105 [31 = (11-(48-105))] $(p.q)\rightarrow(p.--q)$

106 [102 = (105-100)] $\Diamond(p.q)\rightarrow\Diamond(p.--q)$

107 [31 = (104-(106 - 107))] $(p\rightarrow p)\rightarrow\Diamond(p.--p)$

108 [31 = (107-(13 - 108))] $(p\rightarrow p)\rightarrow-(p\rightarrow -p)$ CCl Ax. 12

109 [100=109] $M(p.p)\rightarrow(((p.p)\rightarrow(p.p))\rightarrow(p.p))$

110 [31 = (A9-(109 - 110))] $(p.p)\rightarrow(((p.p)\rightarrow(p.p))\rightarrow(p.p))$

111 [31 = (3 - 111)] $(((p.p)\rightarrow(p.p))\rightarrow(p.P))\rightarrow((p\rightarrow p)\rightarrow(p.p))$

112 [31 = (110 -(111 -112))] $(p\ \ p)\rightarrow((p\rightarrow p)\rightarrow(p.p))$ CCl Axiom 6

113 [9 = (100 -113)] $-((p\rightarrow p)\rightarrow p)\rightarrow -Mp$

114 [48 = (113 - 114)] $(-((p\rightarrow p)\rightarrow p).p)\rightarrow(p.-\Box p)$

115 [48 =(7 - 115)] $(--p.-((p\rightarrow p)\rightarrow p))\rightarrow(-((p\rightarrow p)\rightarrow p).p)$

116 [31 = (115-(114 -116))] $(--p.-((p\rightarrow p)\rightarrow p))\rightarrow(p.-\Box p)$

117. [31 = (34 - (25 - 117))] $(p\rightarrow q)\rightarrow((--p.-r)\rightarrow(q.-r))$

118 [31 = (117-(9 - 118))] $(p\rightarrow q)\rightarrow(-(q.-r)\rightarrow-(--p.-r))$

119 [118 = (116 - (A10- 119)) $-(--(--p.-((p\rightarrow p)\rightarrow p)).-((q\vee q)\rightarrow p))$

120 [119 = 120) $(-p\vee((p\rightarrow p)\rightarrow p))\vee((q\vee q)\rightarrow p)$

121 [120-121) $(-p\vee((p\rightarrow p)\rightarrow p))\vee((p\rightarrow p)\vee(p\rightarrow p))\rightarrow p)$ CCl Axiom 11